

# Scattering length of composite bosons in the 3D BCS-BEC crossover

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We study the zero-temperature grand potential of a three-dimensional superfluid made of ultracold fermionic alkali-metal atoms in the BCS-BEC crossover. In particular, we analyze the zero-point energy of both fermionic single-particle excitations and bosonic collective excitations. The bosonic elementary excitations, which are crucial to obtain a reliable equation of state in the BEC regime, are obtained with a low-momentum expansion up to the fourth order of the quadratic (Gaussian) action of the fluctuating pairing field. By performing a cutoff regularization and renormalization of Gaussian fluctuations, we find that the scattering length  $a_B$  of composite bosons, bound states of fermionic pairs, is given by  $a_B = (2/3) a_F$ , where  $a_F$  is the scattering length of fermions.

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## I. INTRODUCTION

Some years ago the crossover from the weakly-paired Bardeen-Cooper-Schrieffer (BCS) state to the Bose-Einstein condensate (BEC) of molecular dimers was experimentally achieved using ultracold fermionic alkali-metal atoms in a three-dimensional (3D) configuration [1–3]. These remarkable experimental results have been investigated by several theoretical groups (for a recent review see [4]). In a recent experiment the BCS-BEC crossover has been analyzed also in a quasi-2D atomic gas [5], where the experimental data are not in full agreement with theoretical predictions [6, 7]. In a very recent paper [8] we have studied Gaussian fluctuations of the zero-temperature attractive Fermi gas in the 2D BCS-BEC crossover showing that they are crucial to get a reliable equation of state in the BEC regime of composite bosons, bound states of fermionic pairs. In particular, performing dimensional regularization and renormalization-group analysis, we have found the remarkable analytical formula  $a_B = a_F/(2^{1/2}e^{1/4})$  which connects the scattering length  $a_B$  of composite bosons to the scattering length  $a_F$  of fermions [8].

In this paper we investigate the relationship between  $a_B$  and  $a_F$  in the 3D BCS-BEC crossover. We study the low-momentum expansion, up to the fourth order, of the quadratic action of the fluctuating pairing field finding that it gives an ultraviolet divergent contribution of the Gaussian fluctuations to the grand potential. We perform a cutoff regularization of this zero-point energy in the BEC regime and obtain the simple but meaningful analytical formula  $a_B = (2/3) a_F$ . This result is in good agreement with other beyond-mean-field theoretical predictions:  $a_B \simeq 0.75 a_F$  based on a diagrammatic approach [9],  $a_B \simeq 0.60 a_F$  derived from a four-body analysis [10] and also from Monte Carlo simulations [11], and

$a_B \simeq 0.55 a_F$  obtained with convergence factors [12, 13]. We stress, however, that our result is fully analytical, contrary to all other beyond-mean-field predictions [9–13], and it is based on a transparent cutoff regularization and subsequent renormalization of bare physical parameters.

## II. THE MODEL

We consider a three-dimensional Fermi gas of ultracold and dilute two-spin-component neutral atoms. The atomic fermions are described in the path integral formalism by the complex Grassmann fields  $\psi_s(\mathbf{r}, \tau)$ ,  $\bar{\psi}_s(\mathbf{r}, \tau)$  with spin  $s = (\uparrow, \downarrow)$  [14, 15]. The partition function  $\mathcal{Z}$  of the uniform system at temperature  $T$ , in a three-dimensional box of volume  $V$ , and with chemical potential  $\mu$  can be written as

$$\mathcal{Z} = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \exp \left\{ -\frac{1}{\hbar} S \right\}, \quad (1)$$

where

$$S = \int_0^{\hbar\beta} d\tau \int_V d^3\mathbf{r} \mathcal{L} \quad (2)$$

is the Euclidean action functional and  $\mathcal{L}$  is the Euclidean Lagrangian density, given by

$$\mathcal{L} = \bar{\psi}_s \left[ \hbar \partial_\tau - \frac{\hbar^2}{2m} \nabla^2 - \mu \right] \psi_s + g \bar{\psi}_\uparrow \bar{\psi}_\downarrow \psi_\downarrow \psi_\uparrow \quad (3)$$

where  $g$  is the strength of the s-wave inter-atomic coupling ( $g < 0$  in the BCS regime) [14, 15]. Summation over the repeated index  $s$  in the Lagrangian is meant and  $\beta \equiv 1/(k_B T)$  with  $k_B$  Boltzmann's constant.

The Lagrangian density  $\mathcal{L}$ , which is quartic in the fermionic fields, can be rewritten as a quadratic form by introducing the auxiliary complex scalar field  $\Delta(\mathbf{r}, \tau)$  so that [14, 15]:

$$\mathcal{Z} = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \mathcal{D}[\Delta, \bar{\Delta}] \exp \left\{ -\frac{S_e(\psi_s, \bar{\psi}_s, \Delta, \bar{\Delta})}{\hbar} \right\}, \quad (4)$$

where

$$S_e(\psi_s, \bar{\psi}_s, \Delta, \bar{\Delta}) = \int_0^{\hbar\beta} d\tau \int_V d^3\mathbf{r} \mathcal{L}_e(\psi_s, \bar{\psi}_s, \Delta, \bar{\Delta}) \quad (5)$$

and the effective Euclidean Lagrangian density  $\mathcal{L}_e(\psi_s, \bar{\psi}_s, \Delta, \bar{\Delta})$  reads

$$\mathcal{L}_e = \bar{\psi}_s \left[ \hbar \partial_\tau - \frac{\hbar^2}{2m} \nabla^2 - \mu \right] \psi_s + \bar{\Delta} \psi_\downarrow \psi_\uparrow + \Delta \bar{\psi}_\uparrow \bar{\psi}_\downarrow - \frac{|\Delta|^2}{g}. \quad (6)$$

In this paper we investigate the effect of fluctuations of the gap field  $\Delta(\mathbf{r}, t)$  around its mean-field value  $\Delta_0$  which may be taken to be real. For this reason we set

$$\Delta(\mathbf{r}, \tau) = \Delta_0 + \eta(\mathbf{r}, \tau), \quad (7)$$

where  $\eta(\mathbf{r}, \tau)$  is the complex paring field of bosonic fluctuations [14–16].

### III. MEAN-FIELD

By setting  $\eta(\mathbf{r}, t) = 0$ , and integrating over the fermionic fields  $\psi_s(\mathbf{r}, t)$  and  $\bar{\psi}_s(\mathbf{r}, t)$  one obtains the mean-field (saddle-point and fermionic single-particle) partition function [14–16]

$$\mathcal{Z}_{mf} = \exp \left\{ -\frac{S_{mf}}{\hbar} \right\} = \exp \{ -\beta \Omega_{mf} \}, \quad (8)$$

where

$$\begin{aligned} \frac{S_{mf}}{\hbar} &= -Tr[\ln(G_0^{-1})] - \beta V \frac{\Delta_0^2}{g} \\ &= -\sum_{\mathbf{k}} [2 \ln(2 \cosh(\beta E_{sp}(k)/2)) - \beta(\epsilon_k - \mu)] \\ &\quad - \beta V \frac{\Delta_0^2}{g}, \end{aligned} \quad (9)$$

with  $\epsilon_k = \hbar^2 k^2 / (2m)$ ,

$$G_0^{-1} = \begin{pmatrix} \hbar \partial_\tau - \frac{\hbar^2}{2m} \nabla^2 - \mu & \Delta_0 \\ \Delta_0 & \hbar \partial_\tau + \frac{\hbar^2}{2m} \nabla^2 + \mu \end{pmatrix} \quad (10)$$

the inverse mean-field Green function, and

$$E_{sp}(k) = \sqrt{(\epsilon_k - \mu)^2 + \Delta_0^2} \quad (11)$$

the energy of the fermionic single-particle elementary excitations.

At zero temperature ( $T = 0$ , i.e.  $\beta \rightarrow +\infty$ ) the mean-field grand potential  $\Omega_{mf}$  becomes

$$\Omega_{mf} = -\sum_{\mathbf{k}} (E_{sp}(k) - \epsilon_k + \mu) - V \frac{\Delta_0^2}{g}. \quad (12)$$

The constant, uniform and real gap parameter  $\Delta_0$  is obtained by minimizing  $\Omega_{mf}$  with respect to  $\Delta_0$ , namely

$$\left( \frac{\partial \Omega_{mf}}{\partial \Delta_0} \right)_\mu = 0. \quad (13)$$

In this way one gets the familiar gap equation

$$-\frac{1}{g} = \frac{1}{V} \sum_{|\mathbf{k}| < \Lambda} \frac{1}{2E_{sp}(k)}, \quad (14)$$

where the ultraviolet cutoff  $\Lambda$  is introduced to avoid the divergence of the right side of Eq. (14) in the continuum limit  $\sum_{\mathbf{k}} \rightarrow V \int d^3\mathbf{k} / (2\pi)^3$ . One can express the bare interaction strength  $g$  in terms of the physical s-wave scattering length  $a_F$  of fermions through [17, 18, 20]

$$-\frac{m}{4\pi\hbar^2 a_F} = -\frac{1}{g} - \frac{1}{V} \sum_{|\mathbf{k}| < \Lambda} \frac{1}{2\epsilon_k}. \quad (15)$$

After integration over momenta Eq. (15) reads

$$-\frac{m}{4\pi\hbar^2 a_F} = -\frac{1}{g} - \frac{m}{2\pi^2\hbar^2} \Lambda. \quad (16)$$

For this expression one deduces that while  $g$  is always negative  $a_F$  can change sign. In particular, in the weak-coupling BCS limit, where  $g \rightarrow 0^-$ , the first term on the right of Eq. (16) dominates and  $a_F = mg / (4\pi\hbar^2) \rightarrow 0^-$ . In the strong-coupling BEC limit, where  $g \rightarrow -\infty$ , the second term on the right of Eq. (16) dominates and  $a_F = \pi / (2\Lambda) \rightarrow 0^+$  when  $\Lambda$  is sent to infinity [20].

Inserting Eq. (15) into Eq. (14) we obtain the regularized gap equation

$$-\frac{m}{4\pi\hbar^2 a_F} = \frac{1}{V} \sum_{|\mathbf{k}| < \Lambda} \left( \frac{1}{2E_{sp}(k)} - \frac{1}{2\epsilon_k} \right), \quad (17)$$

where one can safely take the limit  $\Lambda \rightarrow +\infty$ , finding the energy gap  $\Delta_0$  as a function of the chemical potential  $\mu$  and the scattering length  $a_F$ . We stress that in the BCS limit, where  $a_F \rightarrow 0^-$ , the chemical potential  $\mu$  is positive and  $\mu/\Delta_0 \rightarrow +\infty$ . At unitarity, where  $a_F \rightarrow \pm\infty$ , Eq. (17) gives  $\mu/\Delta_0 = 0.8604$  showing that the chemical potential  $\mu$  is still positive. In the BEC regime, where  $a_F \rightarrow 0^+$ , the chemical potential becomes negative and it is given by

$$\mu = -\frac{\hbar^2}{2ma_F^2} + \frac{1}{4} \frac{ma_F^2}{\hbar^2} \Delta_0^2, \quad (18)$$

while  $\mu/\Delta_0 \rightarrow -\infty$ . Notice that  $\epsilon_B = \hbar^2 / (ma_F^2)$  is the binding energy of the atomic dimers (composite bosons)

which are formed at unitarity [17, 18, 20] and clearly  $\mu = -\epsilon_B/2$  in the deep BEC limit.

By using Eq. (15) (with  $\Lambda \rightarrow +\infty$ ) with Eq. (12) we obtain the zero-temperature regularized mean-field grand potential of fermionic single-particle excitations

$$\Omega_{mf} = - \sum_{\mathbf{k}} \left( E_{sp}(\mathbf{k}) - \epsilon_{\mathbf{k}} + \mu - \frac{\Delta_0^2}{2\epsilon_{\mathbf{k}}} \right) - V \frac{m}{4\pi\hbar^2 a_F} \Delta_0^2. \quad (19)$$

In the BCS limit, where  $a_F \rightarrow 0^-$ , the energy gap  $\Delta_0$  goes to zero and the mean-field grand potential becomes that of a non-interacting Fermi gas, namely

$$\Omega_{mf} = -V \frac{2}{15\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \mu^{5/2}. \quad (20)$$

In the BEC limit, where  $a_F \rightarrow 0^+$ , the mean-field grand potential reads [17]

$$\Omega_{mf} = -V \frac{1}{256\pi} \left( \frac{2m}{\hbar^2} \right)^{3/2} \frac{\Delta_0^4}{|\mu|^{3/2}}. \quad (21)$$

We point out that in this limit both  $\Delta_0$  and  $|\mu|$  ( $\mu < 0$ ) diverge but  $|\mu|$  diverges faster and the grand potential  $\Omega_{mf}$  goes to zero.

#### IV. GAUSSIAN QUANTUM FLUCTUATIONS

We now consider the effect of quantum fluctuations, i.e. in Eq. (7) we allow  $\eta(\mathbf{r}, t) \neq 0$ . Expanding the effective action  $S_e(\psi_s, \bar{\psi}_s, \Delta, \bar{\Delta})$  of Eq. (5) around  $\Delta_0$  up to the quadratic (Gaussian) order in  $\eta(\mathbf{r}, t)$  and  $\bar{\eta}(\mathbf{r}, t)$  one finds

$$Z = Z_{mf} \int \mathcal{D}[\eta, \bar{\eta}] \exp \left\{ -\frac{S_g(\eta, \bar{\eta})}{\hbar} \right\}, \quad (22)$$

where

$$S_g(\eta, \bar{\eta}) = \frac{1}{2} \sum_q (\bar{\eta}(q), \eta(-q)) \mathbf{M}(q) \begin{pmatrix} \eta(q) \\ \bar{\eta}(-q) \end{pmatrix} \quad (23)$$

is the Gaussian action of fluctuations in the reciprocal space with  $q = (\mathbf{q}, i\nu_m)$  the 4-vector denoting the momenta  $\mathbf{q}$  and Bose Matsubara frequencies  $\nu_m = 2\pi m/\beta$ . The  $2 \times 2$  matrix  $\mathbf{M}(q)$  is the inverse fluctuation propagator, whose non trivial dependence on  $q$  can be found in Ref. [13, 19]. The energy  $E_{col}(q)$  of the bosonic collective excitations can be extracted from  $M(q)$  [13, 19–22] and it is given by

$$E_{col}(q) = \sqrt{\epsilon_q (\lambda \epsilon_q + 2 m c_s^2)} \quad (24)$$

where  $\epsilon_q = \hbar^2 q^2/(2m)$  is the free-particle energy,  $\lambda$  takes into account the first correction to the familiar low-momentum phonon dispersion  $E_{col}(q) \simeq c_s \hbar q$ ,  $c_s$  being

the sound velocity. Both  $\lambda$  and  $c_s$  depend on the chemical potential  $\mu$  and the energy gap  $\Delta_0$ . In particular, one finds [18, 22]

$$\lambda = \begin{cases} -\frac{128}{135} \frac{\mu^2}{\Delta_0^2} & \text{BCS limit} \\ \frac{1}{8} & \text{unitarity} \\ \frac{1}{4} & \text{BEC limit} \end{cases} \quad (25)$$

and

$$m c_s^2 = \begin{cases} \frac{2}{3} \mu & \text{BCS limit} \\ \frac{3}{2} \mu & \text{unitarity} \\ \frac{1}{8} \frac{\Delta_0^2}{|\mu|} & \text{BEC limit} \end{cases} \quad (26)$$

Integrating over the bosonic fields  $\eta(q)$  and  $\bar{\eta}(q)$  in Eq. (22), at zero temperature we find the grand potential

$$\Omega = - \lim_{\beta \rightarrow +\infty} \frac{1}{\beta} \ln(Z) = \Omega_{mf} + \Omega_g \quad (27)$$

where  $\Omega_{mf}$  is given by Eq. (19), while  $\Omega_g$  reads

$$\Omega_g = \frac{1}{2} \sum_{|\mathbf{q}| < \Lambda} E_{col}(q). \quad (28)$$

This is the zero-point energy of bosonic collective excitations [17, 19, 20]. Also here the ultraviolet cutoff  $\Lambda$  is introduced to avoid the divergence in the continuum limit  $\sum_{\mathbf{q}} \rightarrow V \int d^3\mathbf{q}/(2\pi)^3$ .

#### V. SCATTERING LENGTH OF COMPOSITE BOSONS IN THE BEC LIMIT

Expanding Eq. (28) in powers of  $\Lambda$  [20] we find

$$\frac{\Omega_g}{V} = \frac{\hbar^2}{40\pi^2 m \lambda^{1/4}} \Lambda^5 + \frac{m c_s^2}{12\pi^2 \lambda^{1/2}} \Lambda^3 - \frac{m^3 c_s^4}{4\pi^2 \lambda^{3/2}} \Lambda + \frac{8m^4 c_s^5}{15\pi^2 \hbar^2 \lambda^2} + O\left(\frac{1}{\Lambda}\right). \quad (29)$$

In this equation the first two terms are truly divergent; the third term, despite being  $\propto \Lambda$  is indeed convergent, as we are going to show shortly and the fourth term is subleading in the BEC limit.

We then want to regularize the  $\Lambda^5$  and  $\Lambda^3$  terms and this can be done by redefining the bare parameters in the mean-field grand potential in eq. (21), introducing their renormalized counterparts. One then finds, as shown in detail in Appendix , that the following regularization pattern removes the divergencies at the leading order:

$$\begin{cases} \Delta_0 \longrightarrow \Delta_0 + \frac{16\sqrt{2}}{5} \frac{|\mu|^{\frac{3}{2}}}{\Delta_0^3} \left( \frac{\hbar^2}{2m} \right)^{\frac{5}{2}} \Lambda^5 \\ |\mu| \longrightarrow |\mu| - \frac{128}{9\pi^3} \frac{|\mu|^{\frac{1}{2}}}{\Delta_0^2} \left( \frac{\hbar^2}{2m} \right)^{\frac{5}{2}} \Lambda^5 \end{cases} \quad (30)$$

having used the deep-BEC relations in Eqs. (25), (26) and, as previously discussed, the fact that in the strong-coupling BEC limit the cutoff  $\Lambda$  can be obtained from

Eq. (16) and it reads

$$\Lambda = \frac{\pi}{2a_F}. \quad (31)$$

We now analyze the leading convergent contribution, i.e. the  $\Lambda$  term of Eq. (29); even though it seems divergent, being proportional to  $\Lambda$ , actually it is not because in the BEC limit  $c_s^4$  goes to zero faster than  $1/\Lambda$ .

Using Eqs. (18), (25), (26), (31) in the deep BEC regime we have  $|\mu| = \hbar^2/(2ma_F^2)$ ,  $\lambda = 1/4$ ,  $mc_s^2 = \Delta_0^2/(8|\mu|)$ , and the  $\Lambda$  term of Eq. (29) becomes

$$\Omega_g = -V \frac{\alpha}{256\pi} \left( \frac{2m}{\hbar^2} \right)^{3/2} \frac{\Delta_0^4}{|\mu|^{3/2}}, \quad (32)$$

with  $\alpha = 2$ . It is important to stress that Eq. (32) is formally the same formula found by Hu, Liu, and Drummond [12] and also by Diener, Sensarma, and Randeria [13] by using a different regularization procedure based on convergence factors. They have determined numerically the parameter  $\alpha$  finding respectively  $\alpha = 2.5$  [12] and  $\alpha = 2.61$  [13], while here we derive  $\alpha = 2$  analytically.

In the BEC limit, taking into account Eqs. (21) and (32) the total grand potential is given by

$$\Omega = \Omega_{mf} + \Omega_g = -V \frac{(1+\alpha)}{256\pi} \left( \frac{2m}{\hbar^2} \right)^{3/2} \frac{\Delta_0^4}{|\mu|^{3/2}}, \quad (33)$$

with  $(1+\alpha) = (1+2) = 3$ . Here the grand potential  $\Omega$  depends explicitly on both  $\mu$  and  $\Delta_0$ . Consequently, the total number  $N$  of fermions must be calculated as follows

$$N = - \left( \frac{\partial \Omega}{\partial \mu} \right)_{V, \Delta_0} - \left( \frac{\partial \Omega}{\partial \Delta_0} \right)_{V, \mu} \frac{\partial \Delta_0}{\partial \mu}, \quad (34)$$

and the number density  $n = N/V$  reads

$$n = \frac{(1+\alpha)}{16\pi} \left( \frac{2m}{\hbar^2} \right)^{3/2} \frac{\Delta_0^2}{|\mu|^{1/2}}. \quad (35)$$

Notice that in Eq. (34) the second term accounts for a variation of the order parameter  $\Delta_0$  as a function of  $\mu$  instead of considering it as an independent variable. The same procedure was adopted in Refs. [12, 13], while in the well-known Nozieres-Schmitt-Rink scheme this term is absent. As stressed in Refs. [12, 13], this second term cannot be neglected as it becomes more and more relevant in the BEC regime we consider here, becoming of the same order of the first term.

Eq. (35) shows that, at fixed number density  $n$ , in the BEC limit, where both  $|\mu|$  and  $\Delta_0$  go to infinity, one has  $|\mu| \sim \Delta_0^4$  and, from Eq. (33) it follows that  $\Omega$  goes to zero. To obtain Eq. (35) we have used Eq. (34) but also Eq. (18), which immediately gives  $\partial \Delta_0 / \partial \mu = 2\hbar^2/(ma_F^2 \Delta_0) \simeq 4|\mu|/\Delta_0$ . Taking into account Eq. (35), the equation (18) for the chemical potential in the BEC limit can be rewritten as

$$\mu = -\frac{\hbar^2}{2ma_F^2} + \frac{\pi \hbar^2}{m} \frac{a_F}{(1+\alpha)} n, \quad (36)$$

where the second term is half of the chemical potential  $\mu_B = 4\pi \hbar^2 a_B n_B / m_B$  of composite bosons of mass  $m_B = 2m$ , density  $n_B = n/2$ , and boson-boson scattering length

$$a_B = \frac{2}{(1+\alpha)} a_F = \frac{2}{3} a_F. \quad (37)$$

This result is in good agreement with other beyond-mean-field theoretical predictions:  $a_B \simeq 0.75a_F$  of Pieri and Strinati [9],  $a_B \simeq 0.60a_F$  of Petrov, Salomon and Shlyapnikov [10] (and also Astrakharchik, Boronat, Casulleras, and S. Giorgini [11]), and  $a_B \simeq 0.55a_F$  of Hu, Liu and Drummond [12] (and also Diener, Sensarma and Randeria [13]). On the other hand the mean-field result is quite different, namely  $a_B = 2a_F$  [16]. As stressed in the introduction, the beauty of our result, Eq. (37), is that it is obtained fully analytically, contrary to all other beyond-mean-field predictions [9, 10, 12, 13].

## VI. CONCLUSIONS

Starting from a theory of attractive fermions, performing cutoff regularization plus renormalization of Gaussian fluctuations, we have obtained a remarkable formula between the scattering length  $a_B$  of composite bosons and the scattering length  $a_F$  of fermions. This formula,  $a_B = (2/3)a_F$ , is based on the two-body scattering theory which gives in the deep BEC regime an explicit relationship between the cutoff parameter and the s-wave scattering length. Our approach, limited to the quartic term in the low-momentum expansion of bosonic collective excitations, is fully reliable in the BEC regime but it cannot describe the entire 3D BCS-BEC crossover. In fact in the BCS region, where the chemical potential  $\mu$  is positive, the sign of the coefficient  $\lambda$  of the collective spectrum is negative (pair instability) and further terms must be included in the momentum expansion.

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## Appendix: Renormalization in the BEC limit

By using Eqs. (25), (26) and (31) we rewrite the divergent terms in the Gaussian grand potential in Eq. (29) as a function of  $\Delta_0$  and  $\mu$

$$\Omega_g^{(\Lambda^3)} = V \frac{\pi}{384} \left( \frac{2m}{\hbar^2} \right)^{\frac{3}{2}} \Delta_0^2 |\mu|^{\frac{1}{2}} \quad (A.1)$$

$$\Omega_g^{(\Lambda^5)} = V \frac{\sqrt{2}\pi^3}{640} \left( \frac{2m}{\hbar^2} \right)^{\frac{3}{2}} |\mu|^{\frac{5}{2}} \quad (\text{A.2})$$

The tree-level mean-field contribution in Eq. (21) is a function of the bare parameters  $\Delta_0$  and  $|\mu|$ , which we now renormalize as follows:

$$\begin{cases} \Delta_0 \longrightarrow \Delta_R = \Delta_0 + \delta\Delta \\ |\mu| \longrightarrow |\mu|_R = |\mu| + \delta\mu \end{cases} \quad (\text{A.3})$$

$$\Omega_{mf}^R = -V \frac{1}{256\pi} \left( \frac{2m}{\hbar^2} \right)^{3/2} \frac{\Delta_R^4}{|\mu|_R^{\frac{3}{2}}} \approx -V \frac{1}{256\pi} \left( \frac{2m}{\hbar^2} \right)^{3/2} \left( \frac{\Delta_0^4}{|\mu|^{\frac{3}{2}}} + 4 \frac{\Delta_0^3}{|\mu|^{\frac{3}{2}}} \delta\Delta - \frac{3}{2} \frac{\Delta_0^4}{|\mu|^{\frac{5}{2}}} \delta\mu \right) \quad (\text{A.4})$$

and in order to find the renormalization pattern in Eq. (30) we simply equate the new terms in the renormalized grand potential in Eq. (A.4) to minus each divergent term, i.e.:

$$\frac{3}{2} V \frac{1}{256\pi} \left( \frac{2m}{\hbar^2} \right)^{3/2} \frac{\Delta_0^4}{|\mu|^{\frac{3}{2}}} \delta\mu = -\Omega_g^{(\Lambda^3)} \quad (\text{A.5})$$

for a suitable choice of the counter-terms. The renormalized mean-field grand potential now reads, to the leading order:

$$-4V \frac{1}{256\pi} \left( \frac{2m}{\hbar^2} \right)^{3/2} \frac{\Delta_0^3}{|\mu|^{\frac{3}{2}}} \delta\Delta = -\Omega_g^{(\Lambda^5)} \quad (\text{A.6})$$

solving for the counter-terms one obtains  $\delta\mu = -\frac{4\pi^2|\mu|^3}{9\Delta_0^2}$  and  $\delta\Delta = \frac{\pi^4|\mu|^4}{5\sqrt{2}\Delta_0^3}$ . Eq. (30) readily follows reinstating  $\Lambda$  by means of Eqs. (25), (26) and (31).

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